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Soliton solutions for a spin chain with an easy plane and the method of Riemann problem with zeros

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Abstract. With the procedure to solve explicitly the equations of the Riemann matrix with poles, multi-soliton solutions to the Landau–Lifshitz (L–L) equation for a spin chain with an easy plane are found formally, and a single soliton solution is given explicitly.

1. Introduction

The Landau–Lifshitz equation for a spin chain with an easy plane describes, for example, the properties of $CsNiF_3$ and has attracted the attention of many authors in the past two decades [1, 2], but has not been solved exactly by any means attempted [3]. Recently, an exact single soliton solution to the equation by means of the method of the Darboux transformation matrix has been reported [3]. The single soliton solution essentially depends on two parameters. These two parameters, namely two velocities, describe a spin configuration, deviating from homogeneous magnetization. The centre of inhomogeneity moves with a constant velocity, while the shape of the soliton (the direction of magnetization in its centre) also changes with another velocity. This feature does not appear in the single soliton solutions of other nonlinear equations solved. Hence it is expected that the method can be developed in detail to find multi-soliton solutions. In [4], a system of linear algebraic equations was derived that yields multi-soliton solutions, in principle, but was, in general, hard to solve explicitly.

It is easily seen that the structure of these linear algebraic equations is the same as that of the equations of a Riemann matrix with poles in the method of the Riemann problem with zeros [5, 6]. As is known, the equations of a Riemann matrix with poles were merely solved for the simplest cases of one or two poles but not for a general number of poles.

In this paper, a particular procedure that leads to explicit solutions of the Riemann matrix with poles is developed and is applied to the L–L equation with an easy plane. It is given that these formulae are sufficient for calculation of multi-soliton solutions of the L–L equation with an easy plane if they are required.

The equations of a Riemann matrix with poles are the same as those deduced from the expansion of the partial fraction of the transformation matrix, although their starting points are different from each other. In this paper, the equations are simply derived from the later point of view. A procedure based upon a formula of the inverse of a particular Q-matrix is developed, and, by means of it, solutions of the equations of a Riemann matrix with poles are expressed in forms which can be evaluated explicitly using the well known Binet–Cauchy formula. The L–L equation for a spin chain with an easy plane is then solved explicitly using this procedure.

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2. The L–L equation with an easy plane

The L–L equation for a spin chain with an easy plane is

$$\boldsymbol{S}_t = \boldsymbol{S} \times \boldsymbol{S}_{xx} + \boldsymbol{S} \times J\boldsymbol{S} \qquad |\boldsymbol{S}| = 1 \tag{1}$$

where the diagonal matrix J

$$J = \mathrm{diag}(0, 0, -16\rho^2) \tag{2}$$

characterizes the easy plane, the 12-plane. Here ρ is a positive constant and 16 is introduced for latter convenience. Its Lax pair is given by Sklyanin [7] by setting the present J,

$$L = -\mathrm{i}\mu S_3\sigma_3 - \mathrm{i}\lambda(S_1\sigma_1 + S_2\sigma_2) \tag{3}$$

$$\begin{aligned} M = {} & \mathrm{i}2\lambda^2 S_3\sigma_3 + \mathrm{i}2\lambda\mu(S_1\sigma_1 + S_2\sigma_2) - \mathrm{i}\lambda(S_2S_{3x} - S_3S_{2x})\sigma_1 \\ & -\mathrm{i}\lambda(S_3S_{1x} - S_1S_{3x})\sigma_2 - \mathrm{i}\mu(S_1S_{2x} - S_2S_{1x})\sigma_3 \end{aligned} \tag{4}$$

where two parameters μ and λ satisfy

$$\mu^2 = \lambda^2 + 4\rho^2. \tag{5}$$

If one is taken as an independent parameter, the other is its double-valued function. To avoid complexity due to this, one can introduce an affine parameter ζ as follows:

$$\mu = \zeta + \rho^2\zeta^{-1} \qquad \lambda = \zeta - \rho^2\zeta^{-1}. \tag{6}$$

However, as we shall see later, in the development of the present method, it is reasonable to introduce an auxiliary parameter k as follows:

$$\mu = 2\rho\frac{k + k^{-1}}{k - k^{-1}} \qquad \lambda = 2\rho\frac{2}{k - k^{-1}}. \tag{7}$$

The Lax equations are

$$\partial_x F(k) = L(k)F(k) \qquad \partial_t F(k) = M(k)F(k) \tag{8}$$

from now on we shall drop the arguments x and t unless necessary.

Since the 12-plane is the easy plane, the asymptotic spin must lie on it, we can choose it along the 1-axis,

$$\boldsymbol{S} = \boldsymbol{S}_0 = (1, 0, 0) \tag{9}$$

which is the simplest solution of (1). To the spin along the 1-axis the corresponding Jost solution of (8) is then

$$F_0(k) = \mathrm{e}^{-\mathrm{i}\lambda(x - 2\mu t)\sigma_1}. \tag{10}$$

3. Transformation matrix

We define the Jost solutions $F_N(k)$ by a transformation matrix $G_N(k)$,

$$F_N(k) = G_N(k)F_0(k) \tag{11}$$

where $G_N(k)$ is meromorphic, that is, it may have poles, but has no other singularity. The meaning of the suffix N shall be explained in the following. The properties of $G_N(k)$ and the relation to the solution $\boldsymbol{S}$ of (1) will be determined.

It is obvious that

$$\mu(-\bar{k}) = \overline{\mu(k)} \qquad \lambda(-\bar{k}) = -\overline{\lambda(k)} \tag{12}$$

and then

$$L(-\bar{k}) = \sigma_1 \overline{L(k)} \sigma_1 \qquad M(-\bar{k}) = \sigma_1 \overline{M(k)} \sigma_1. \tag{13}$$

From (10) we can see that

$$F_0(-\bar{k}) = \sigma_1 \overline{F_0(k)} \sigma_1. \tag{14}$$

Hence we have

$$F_N(-\bar{k}) = \sigma_1 \overline{F_N(k)} \sigma_1 \qquad G_N(-\bar{k}) = \sigma_1 \overline{G_N(k)} \sigma_1. \tag{15}$$

Suppose that k_n is a simple pole of $G_N(k)$, then from (15) $-\bar{k}_n$ is also a pole of $G_N(k)$. We denote it as $k_{\breve{n}} = -\bar{k}_n$. The suffix N means that $G_N(k)$ has N pairs of simple poles. We can write

$$G_N(k) = K_N H_N(k) \tag{16}$$

where

$$H_N(k) = I + \sum_{n=1}^{2N} \frac{1}{k - k_n} A_n \tag{17}$$

and K_N is a 2×2 matrix independent of k to be determined. We agree on

$$k_{\breve{n}} = \begin{cases} k_{N+n} & \text{if } n \leqslant N \\ k_{n-N} & \text{if } n > N. \end{cases} \tag{18}$$

Since in the limit of $|k| \to \infty$, $L(k)$ and $M(k)$ do not tend to zero, and thus $G_N(k) \not\to I$. Hence K_N differs from I. From (15) we have

$$K_N = \sigma_1 \overline{K}_N \sigma_1 \qquad H_N(-\bar{k}) = \sigma_1 \overline{H_N(k)} \sigma_1 \tag{19}$$

and then

$$A_{\breve{n}} = -\sigma_1 \overline{A}_n \sigma_1. \tag{20}$$

From (3), (4) and (10) we can see that the Lax pair has anti-Hermitian evolution properties,

$$L(k) = -L^\dagger(\bar{k}) \qquad M(k) = -M^\dagger(\bar{k}) \tag{21}$$

and

$$F_0^{-1}(k) = F_0^\dagger(\bar{k}). \tag{22}$$

Hence we have

$$F_N^{-1}(k) = F_N^\dagger(\bar{k}) \qquad G_N^{-1}(k) = G_N^\dagger(\bar{k}). \tag{23}$$

From (16) we obtain

$$G_N^{-1}(k) = H_N^{-1}(k) K_N^{-1}. \tag{24}$$

On account of the second equation of (23), we obtain

$$K_N^{-1} = K_N^\dagger \tag{25}$$

and

$$H_N^{-1}(k) = H_N^\dagger(\bar{k}) \tag{26}$$

where

$$H_N^\dagger(\bar{k}) = I + \sum_{n=1}^{N} \frac{1}{k - \bar{k}_n} A_n^\dagger. \tag{27}$$

4. Equations of the Riemann matrix with poles

Since

$$G_N(k)G_N^{-1}(k) = G_N^{-1}(k)G_N(k) = I \tag{28}$$

it has no poles, i.e.

$$A_n G_N^{-1}(k_n) = 0 \tag{29}$$

i.e.

$$A_n\left(I + \sum_{m=1}^{2N} \frac{1}{k_n - \bar{k}_m} A_m^{\dagger}\right) = 0. \tag{30}$$

One can write

$$A_n = \begin{pmatrix} \delta_n \\ \gamma_n \end{pmatrix} (\beta_n \alpha_n). \tag{31}$$

Substituting it into (30) we obtain a system of linear equations,

$$\beta_n + \sum_{m=1}^{2N} \frac{1}{k_n - \bar{k}_m} (\beta_n \bar{\beta}_m + \alpha_n \bar{\alpha}_m)\bar{\delta}_m = 0 \tag{32}$$

and

$$\alpha_n + \sum_{m=1}^{2N} \frac{1}{k_n - \bar{k}_m} (\beta_n \bar{\beta}_m + \alpha_n \bar{\alpha}_m)\bar{\gamma}_m = 0. \tag{33}$$

Solving these equations one can express δ_m and γ_m in terms of β_n and α_n. However, except at low values of N, it is hard to obtain explicit solutions. These equations are some of the Riemann matrix with poles. Hence the transformation matrix defined in (11) can be referred to as a Riemann matrix with poles. We shall develop a special procedure to give explicit solutions for the present case but also suitable for some other cases.

5. Determination of β_n and α_n

For particular N, the Lax pairs $L(k)$ and $M(k)$ are expressed as $L_N(k)$ and $M_N(k)$, we have

$$\partial_x F_N(k) = L_N(k)F_N(k). \tag{34}$$

In the limit as $k \to k_n$, on account of (11) we have

$$\partial_x\{K_N A_n F_0(k_n)\} = L_N(k_n)\{K_N A_n F_0(k_n)\}. \tag{35}$$

Since A_n is degenerate, the factor

$$(\beta_n \alpha_n)F_0(k_n) \tag{36}$$

can be taken to be independent of x. From

$$\partial_t F_N(k) = M_N(k)F_N(k) \tag{37}$$

a similar procedure yields that the factor (36) is also independent of t. Hence we find

$$(\beta_n \alpha_n) = (b_n \ 1)F_0^{-1}(k_n) \tag{38}$$

where b_n is a constant.

From (20) we have

$$A_{\breve{n}} = -\begin{pmatrix} \bar{\gamma}_n \\ \bar{\delta}_n \end{pmatrix} (\bar{\alpha}_n \bar{\beta}_n)$$

and also

$$A_{\breve{n}} = \begin{pmatrix} \delta_{\breve{n}} \\ \gamma_{\breve{n}} \end{pmatrix} (\beta_{\breve{n}} \alpha_{\breve{n}}).$$

We can agree on

$$\beta_{\breve{n}} = \mathrm{i}\bar{\alpha}_n \qquad \alpha_{\breve{n}} = \mathrm{i}\bar{\beta}_n \tag{39}$$

and thus

$$\delta_{\breve{n}} = \mathrm{i}\bar{\gamma}_n \qquad \gamma_{\breve{n}} = \mathrm{i}\bar{\delta}_n \qquad b_{\breve{n}} = \bar{b}_n. \tag{40}$$

6. Determination of K_N

From (19) we have

$$(K_N)_{11} = (\overline{K}_N)_{22} \qquad (K_N)_{21} = (\overline{K}_N)_{12}. \tag{41}$$

(25) yields

$$\frac{1}{\det K_N} \begin{pmatrix} (K_N)_{22} & -(K_N)_{12} \\ -(K_N)_{21} & (K_N)_{11} \end{pmatrix} = \begin{pmatrix} (\overline{K}_N)_{11} & (\overline{K}_N)_{21} \\ (\overline{K}_N)_{12} & (\overline{K}_N)_{22} \end{pmatrix}. \tag{42}$$

Comparing these two equations leads to the unimodule of K_N and

$$K_N = \mathrm{e}^{\mathrm{i}\frac{1}{2}\Omega_N \sigma_3} \tag{43}$$

where Ω_N is real and may depend on x and t. From (19) and (25) similar relations hold for $H_N(0)$, and hence $H_N(0)$ is also unimodule and diagonal. Similarly, (25) leads to $H_N(1)$ and $G_N(1)$ are unimodule and

$$H_N(1)_{11} = \overline{H}_N(1)_{22} \qquad H_N(1)_{21} = -\overline{H}_N(1)_{12} \tag{44}$$

$$G_N(1)_{11} = \overline{G}_N(1)_{22} \qquad G_N(1)_{21} = -\overline{G}_N(1)_{12}. \tag{45}$$

From (34) we have

$$G_{Nx}(k) + G_N(k)L_0(k) = L_N(k)G_N(k). \tag{46}$$

As $k \to 1$, we obtain

$$(\boldsymbol{S}_N \cdot \boldsymbol{\sigma}) = G_N(1)\sigma_1 G_N^\dagger(1) \tag{47}$$

where $\boldsymbol{S}_N$ is the solution of (1) corresponding to the Jost solution $F_N(k)$. Similarly, as $k \to -1$, we have

$$-\sigma_3(\boldsymbol{S}_N \cdot \boldsymbol{\sigma})\sigma_3 = G_N(-1)\sigma_1 G_N^\dagger(-1) \tag{48}$$

which is equivalent to (34) on account of the second equation of (15).

We now determine K_N. From (38), as $k \to \infty$, we obtain

$$\partial_x\{K_N\} = -\mathrm{i}2\rho(S_N)_3\sigma_3\{K_N\} \tag{49}$$

as $k \to 0$, we have

$$\partial_x\{K_N H_N(0)\} = \mathrm{i}2\rho(S_N)_3\sigma_3\{K_N H_N(0)\}. \tag{50}$$

These two equations lead to

$$K_N = H_N(0)^{-1/2}. \tag{51}$$

From (43), Ω_N is an additional rotation angle around the 3-axis which does not effect the value of S_3.

7. Expressions for the spin components

Noticing (45) and (51), from (47) we obtain

$$(S_N)_3 = -\overline{G_N(1)}_{11}\overline{G_N(1)}_{21} - G_N(1)_{11}G_N(1)_{21} \tag{52}$$
$$(S_N)_1 - \mathrm{i}(S_N)_2 = -\big(\overline{G_N(1)}_{21}\big)^2 + \big(G_N(1)_{11}\big)^2 \tag{53}$$

or

$$(S_N)_3 = -\overline{H_N(1)}_{11}\overline{H_N(1)}_{21} - H_N(1)_{21}H_N(1)_{11} \tag{54}$$
$$(S_N)_1 - \mathrm{i}(S_N)_2 = (K_N)_{11}^2\{-(\overline{H_N(1)}_{21})^2 + (H_N(1)_{11})^2\}. \tag{55}$$

(55) is

$$(S_N)_1 - \mathrm{i}(S_N)_2 = \overline{H_N(0)}_{11}\{-(\overline{H_N(1)}_{21})^2 + (H_N(1)_{11})^2\}. \tag{56}$$

Hence it is necessary to find $H_N(1)$ and $H_N(0)$.

From (17) we have

$$H_N(1)_{11} = 1 + \sum_{j=1}^{2N} \frac{1}{1-k_j}\delta_j\beta_j \tag{57}$$

$$H_N(1)_{21} = \sum_{j=1}^{2N} \frac{1}{1-k_j}\gamma_j\beta_j \tag{58}$$

$$H_N(0)_{11} = 1 - \sum_{j=1}^{2N} \frac{1}{k_j}\delta_j\beta_j \tag{59}$$

and these must be evaluated.

In what follows it is convenient to introduce

$$p_n = -\mathrm{i}k_n. \tag{60}$$

From (32) and (33) we have

$$\sum_{m=1}^{M} \frac{1}{p_m+\bar{p}_n}(\beta_m\bar{\beta}_n + \alpha_m\bar{\alpha}_n)\delta_m = \mathrm{i}\bar{\beta}_n. \tag{61}$$

and

$$\sum_{m=1}^{M} \frac{1}{p_m+\bar{p}_n}(\beta_m\bar{\beta}_n + \alpha_m\bar{\alpha}_n)\gamma_m = \mathrm{i}\bar{\alpha}_n. \tag{62}$$

Here we write $M = 2N$. Then instead of (57)–(59), what we need to evaluate is

$$H_N(p_0)_{11} = 1 + \mathrm{i}\sum_{j=1}^{M} \frac{1}{\bar{p}_0+p_j}\delta_j\beta_j \qquad H_N(p_0)_{21} = \mathrm{i}\sum_{j=1}^{M} \frac{1}{\bar{p}_0+p_j}\gamma_j\beta_j \tag{63}$$

where $p_0 = -\mathrm{i}k_0, k_0 = 1$, or 0.

8. Solutions of a Riemann matrix with poles

We write introducing diagonal matrices (β), etc, by

$$(\beta) = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_M) \tag{64}$$

and row matrices β, etc, by

$$\beta = (\beta_1 \beta_2 \ldots \beta_M) \tag{65}$$

then (61) and (62) can be rewritten in matrix form,

$$\delta(B + C) = \mathrm{i}\bar{\beta} \qquad \gamma(B + C) = \mathrm{i}\bar{\alpha} \tag{66}$$

where

$$B = (\beta)Q(\bar{\beta}) \qquad C = (\alpha)Q(\bar{\alpha}) \tag{67}$$

and the matrix Q is given in (A.1). Referring to (A.9), we have

$$C^{-1} = ((\alpha)^{-1}(\boldsymbol{a})^{-1}Q(\bar{\boldsymbol{a}})^{-1}(\bar{\alpha})^{-1})^{\mathrm{T}} \tag{68}$$

where $(\boldsymbol{a})$ is given by (A.4) and (A.8).

The formulae can be simplified if

$$C^{-1} = B^{\mathrm{T}}. \tag{69}$$

By comparing (68) and (67) this can be achieved by setting

$$\beta_n \alpha_n = a_n^{-1} \tag{70}$$

where $a_n = -\mathrm{i}\boldsymbol{a}_n$. Here we should note that, from (A.5) and (39), we have

$$\boldsymbol{a}_{\check{n}} = \bar{\boldsymbol{a}}_n \qquad \beta_{\check{n}}\alpha_{\check{n}} = -\bar{\beta}_n\bar{\alpha}_n. \tag{71}$$

Hence on the right-hand side of (70) we must choose a_n^{-1} instead of $\boldsymbol{a}_n^{-1}$.

As we have seen, since only the proportion of β_n and α_n has meaning, we can choose β_n and α_n to satisfy (70). Hence multiplying the first one of (66) by B^{T} from the right, we have

$$\delta(I + BB^{\mathrm{T}}) = \mathrm{i}\bar{\beta}B^{\mathrm{T}}. \tag{72}$$

Then (63) is rewritten as

$$H_N(p_0)_{11} = 1 + \mathrm{i}\delta\beta'^{\mathrm{T}} \qquad H_N(p_0)_{21} = \mathrm{i}\gamma\beta'^{\mathrm{T}} \tag{73}$$

where

$$\beta'_j = \frac{1}{\bar{p}_0 + p_j}\beta_j. \tag{74}$$

By using (72) we obtain

$$H_N(p_0)_{11} = 1 - \bar{\beta}B^{\mathrm{T}}(I + BB^{\mathrm{T}})^{-1}\beta'^{\mathrm{T}} = 1 - \mathrm{Tr}\{(I + BB^{\mathrm{T}})^{-1}\beta'^{\mathrm{T}}\bar{\beta}B^{\mathrm{T}}\} \tag{75}$$

and then

$$H_N(p_0)_{11} = \frac{\det(I + BB^{\mathrm{T}} - \beta'^{\mathrm{T}}\bar{\beta}B^{\mathrm{T}})}{\det(I + BB^{\mathrm{T}})}. \tag{76}$$

We note that

$$\mathrm{i}\sum_{n=1}^{M}\bar{\alpha}_n(B^{\mathrm{T}})_{nm} = \mathrm{i}\sum_{n=1}^{M}\bar{\alpha}_n\beta_m\frac{1}{p_m + \bar{p}_n}\bar{\beta}_n. \tag{77}$$

Referring to (70) and (A.12) it is equal to

$$\mathrm{i}\sum_{n=1}^{M}\frac{1}{\mathrm{i}\bar{a}_n}\frac{1}{p_m+\bar{p}_n}\beta_m=\beta_m. \tag{78}$$

Multiplying the second equation of (66) by B^{T} from the right, and using (78) we obtain

$$\gamma(I+BB^{\mathrm{T}})=\beta. \tag{79}$$

Hence

$$H_N(p_0)_{21}=\mathrm{i}\beta(I+BB^{\mathrm{T}})^{-1}\beta'^{\mathrm{T}}=\mathrm{iTr}\{(I+BB^{\mathrm{T}})^{-1}\beta'^{\mathrm{T}}\beta\} \tag{80}$$

namely

$$H_N(p_0)_{21}=\mathrm{i}\left\{\frac{\det(I+BB^{\mathrm{T}}+\beta'^{\mathrm{T}}\beta)}{\det(I+BB^{\mathrm{T}})}-1\right\}. \tag{81}$$

It is known that expressions (76) and (81) can be evaluated by the known Binet–Cauchy formula.

9. Binet–Cauchy formula

The denominator of (76) and (81) is now denoted by W_N, that is

$$W_N=\det(I+BB^{\mathrm{T}})=1+\sum_{r=1}^{M}\sum_{1\leqslant n_1<\cdots<n_r\leqslant M}\sum_{1\leqslant m_1<\cdots<m_r\leqslant M}\times W(n_1,n_2,\ldots,n_r;m_1,m_2,\ldots,m_r) \tag{82}$$

where

$$W(n_1,n_2,\ldots,n_r;m_1,m_2,\ldots,m_r)=B(n_1,n_2,\ldots,n_r;m_1,m_2,\ldots,m_r)^2$$
$$=\prod_{n,m}\frac{\beta_n^2\bar{\beta}_m^2}{(p_n+\bar{p}_m)^2}\prod_{n<n',m<m'}(p_n-p_{n'})^2(\bar{p}_m-\bar{p}_{m'})^2 \tag{83}$$

and where n, n', m and m' satisfy (A.3).

The numerator of (76) can be rewritten as

$$V_N(p_0)=\det(I+BB^{\mathrm{T}}-\beta'^{\mathrm{T}}\bar{\beta}B^{\mathrm{T}})=\det(I+B'B^{\mathrm{T}}) \tag{84}$$

where

$$B'=B-\beta'^{\mathrm{T}}\bar{\beta}. \tag{85}$$

From (74) we have

$$(B')_{nm}=\beta_n\frac{1}{p_n+\bar{p}_m}\frac{\bar{p}_0-\bar{p}_m}{p_n+\bar{p}_0}\bar{\beta}_m. \tag{86}$$

Hence we obtain

$$V_N(p_0)=\det(I+B'B^{\mathrm{T}})=1+\sum_{r=1}^{M}\sum_{1\leqslant n_1<\cdots<n_r\leqslant M}\sum_{1\leqslant m_1<\cdots<m_r\leqslant M}\times V(n_1,n_2,\ldots,n_r;m_1,m_2,\ldots,m_r) \tag{87}$$

where

$$
\begin{aligned}
&V(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r)\\
&\qquad = B'(n_1, \ldots, n_r; m_1, \ldots, m_r)(n_1, \ldots, n_r; m_1, \ldots, m_r)\\
&\qquad = \prod_{n,m} \frac{\bar{p}_0 - \bar{p}_m}{p_n + \bar{p}_0} \frac{\beta_n^2 \bar{\beta}_m^2}{(p_n + \bar{p}_m)^2} \prod_{n<n', m<m'} (p_n - p_{n'})^2 (\bar{p}_m - \bar{k}_{m'})^2
\end{aligned} \tag{88}
$$

and where n, n', m and m' satisfy (A.3).

The numerator of (81) is

$$
U_N(p_0) = \mathrm{i}\{\det(I + BB^{\mathrm{T}} + \beta'^{\mathrm{T}}\beta) - \det(I + BB^{\mathrm{T}})\}. \tag{89}
$$

We can write

$$
I + BB^{\mathrm{T}} + \beta'^{\mathrm{T}}\beta = I + \check{B}'\check{B}^{\mathrm{T}} \tag{90}
$$

where $\check{B}'$ and $\check{B}$ are $M \times (M+1)$ matrices,

$$
\check{B}'_{n0} = \beta'_n \qquad \check{B}_{n0} = \beta_n \qquad \check{B}'_{nm} = \check{B}_{nm} = B_{nm} \qquad n, m = 1, 2, \ldots, M. \tag{91}
$$

By means of the Binet–Cauchy formula, the expansion involves

$$
\begin{aligned}
&\sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant M} \; \sum_{0 \leqslant m_1 < m_2 < \cdots < m_r \leqslant M}\\
&\qquad \times \check{B}'(n_1, \ldots, n_r; m_1, \ldots, m_r) \check{B}(n_1, \ldots, n_r; m_1, \ldots, m_r).
\end{aligned} \tag{92}
$$

It consists of two parts: one is $m_1 \geqslant 1$, the other $m_1 = 0$. The first part is just the same as $\det(I + BB^{\mathrm{T}})$. Hence we obtain

$$
U_N(p_0) = \mathrm{i} \sum_{r=1}^{M} \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant M} \; \sum_{1 \leqslant m_2 < \cdots < m_r \leqslant M} U(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r) \tag{93}
$$

where

$$
\begin{aligned}
&U(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r)\\
&\qquad = \check{B}'(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r) \check{B}(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r)
\end{aligned} \tag{94}
$$

and

$$
\begin{aligned}
&\check{B}'(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r)\\
&\qquad = \prod_{n,m} \frac{\bar{p}_0 - \bar{p}_m}{p_n + \bar{p}_0} \frac{\beta_n \bar{\beta}_m}{p_n + \bar{p}_m} \prod_{n<n', m<m'} (p_n - p_{n'})(\bar{p}_m - \bar{p}_{m'})
\end{aligned} \tag{95}
$$

$$
\check{B}(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r) = \prod_{n,m} \frac{\beta_n \bar{\beta}_m}{p_n + \bar{p}_m} \prod_{n<n', m<m'} (p_n - p_{n'})(\bar{p}_m - \bar{p}_{m'}). \tag{96}
$$

The subscripts satisfy

$$
n, n' \in \{n_1, n_2, \ldots, n_r\} \qquad m, m' \in \{m_2, \ldots, m_r\}. \tag{97}
$$

10. Formal expressions of solutions

Equation (10) can be rewritten as

$$F_0(x,k) = U\mathrm{e}^{-\mathrm{i}\kappa(x-2\lambda t)\sigma_3}U^{-1} \tag{98}$$

where

$$U = \tfrac{1}{2}\{I - \mathrm{i}(\sigma_1+\sigma_2+\sigma_3)\}. \tag{99}$$

We can write

$$(b_n 1)U \sim (b'_n 1) \tag{100}$$

where $\sim$ is a proportional symbol. From (38) we can see that

$$(\beta_n \alpha_n) \sim \begin{pmatrix} f_n & 0 \\ 0 & f_n^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} \tag{101}$$

since the last determinant is proportional to U^{-1}. Here

$$f_n^2 = b'_n f_0(k_n)^2 \qquad f_0(k_n) = \mathrm{e}^{\mathrm{i}\lambda_n(x-2\mu_n t)}. \tag{102}$$

Hence we have

$$\frac{\beta_n}{\alpha_n} = \frac{f_n + \mathrm{i}f_n^{-1}}{f_n - \mathrm{i}f_n^{-1}} \equiv q_n^2. \tag{103}$$

From equation (70) together with (103) we obtain

$$\beta_n = (a_n)^{-\frac{1}{2}} q_n \qquad \alpha_n = (a_n)^{-\frac{1}{2}} q_n^{-1}. \tag{104}$$

Hence (83), (88) and (94) are expressed explicitly

$$W(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) = \prod_{n,m} a_n^{-1}\bar{a}_m^{-1} \frac{q_n^2\bar{q}_m^2}{(p_n+\bar{p}_m)^2} \prod_{n<n',m<m'} (p_n - p_{n'})^2(\bar{p}_m - \bar{p}_{m'})^2 \tag{105}$$

$$V(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) = \prod_{n,m} a_n^{-1}\bar{a}_m^{-1} \frac{\bar{p}_0 - \bar{p}_m}{p_n + \bar{p}_0} \frac{q_n^2\bar{q}_m^2}{(p_n+\bar{p}_m)^2} \prod_{n<n',m<m'} (p_n - p_{n'})^2(\bar{p}_m - \bar{p}_{m'})^2 \tag{106}$$

and where n, n', m and m' satisfy (A.3).

$$U(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r) = \prod_{n,m} a_n^{-1}\bar{a}_m^{-1} \frac{\bar{p}_0 - \bar{p}_m}{p_n + \bar{p}_0} \frac{q_n^2\bar{q}_m^2}{(p_n+\bar{p}_m)^2} \prod_{n<n',m<m'} (p_n - p_{n'})^2(\bar{p}_m - \bar{p}_{m'})^2 \tag{107}$$

where n, n', m and m' satisfy (97).

In order to express explicitly the final expressions, we note (34), (35) and (104) and $q_{\check{n}} = \bar{q}_n^{-1}$. Thus (73) are expressed as

$$H_N(p_0)_{11} = \frac{V_N(p_0)}{W_N} \qquad H_N(-\mathrm{i})_{21} = \frac{U_N(-\mathrm{i})}{W_N}. \tag{108}$$

11. Explicit expressions of a single soliton solution

In the case of $N = 1$, we have $k_2 = -\bar{k}_1, \quad p_2 = \bar{p}_1$ and

$$a_1 = -\mathrm{i}\frac{p_1 - \bar{p}_1}{p_1 + \bar{p}_1}\frac{1}{2p_1} = -\bar{a}_2.$$

From (82)–(97), we have

$$W_1 = \frac{4}{|p_1 - \bar{p}_1|^2}(p_1|q_1|^2 - \bar{p}_1|q_1|^2)(\bar{p}_1|q_1|^2 - p_1|q_1|^{-2}). \tag{109}$$

$$V_1(0) = -\frac{4}{|p_1 - \bar{p}_1|^2}(\bar{p}_1|q_1|^2 - p_1|q_1|^{-2})^2. \tag{110}$$

$$V_1(-\mathrm{i}) = \frac{1}{(p_1 + \mathrm{i})(\bar{p}_1 + \mathrm{i})}\frac{4}{|p_1 - \bar{p}_1|^2}(\bar{p}_1|q_1|^2 - p_1|q_1|^{-2}) \times\{(1 + \bar{p}_1^2)p_1|q_1|^2 + (1 + p_1^2)\bar{p}_1|q_1|^{-2}\} \tag{111}$$

and

$$U_1(-\mathrm{i}) = \mathrm{i}\frac{1}{(p_1 + \mathrm{i})(\bar{p}_1 + \mathrm{i})}\frac{4}{|p_1 - \bar{p}_1|^2}(\bar{p}_1^2 - p_1^2)\{-p_1 q_1^2 + \bar{p}_1\bar{q}_1^{-2}\}. \tag{112}$$

We write

$$g_n = f_n + \mathrm{i}f_n^{-1} \qquad h_n = f_n - \mathrm{i}f_n^{-1} \tag{113}$$

then we have

$$W_1 = \frac{4}{(|p_1 - \bar{p}_1|^2)|g_1|^2|h_1|^2}(p_1|g_1|^2 - \bar{p}_1|h_1|^2)(\bar{p}_1|g_1|^2 - p_1|h_1|^2) \tag{114}$$

$$V_1(0) = -\frac{4}{(|p_1 - \bar{p}_1|^2)|g_1|^2|h_1|^2}(\bar{p}_1|g_1|^2 - p_1|h_1|^2)^2 \tag{115}$$

$$V_1(-\mathrm{i}) = \frac{1}{(p_1 + \mathrm{i})(\bar{p}_1 + \mathrm{i})}\frac{4}{(|p_1 - \bar{p}_1|^2)|g_1|^2|h_1|^2} \times(\bar{p}_1|g_1|^2 - p_1|h_1|^2)\{-(1 + \bar{p}_1^2)p_1|g_1|^2 + (1 + p_1^2)\bar{p}_1|h_1|^2\} \tag{116}$$

and

$$U_1(-\mathrm{i}) = \mathrm{i}\frac{1}{(p_1 + \mathrm{i})(\bar{p}_1 + \mathrm{i})}\frac{4}{(|p_1 - \bar{p}_1|^2)|g_1|^2|h_1|^2} \times(\bar{p}_1^2 - p_1^2)(-p_1|g_1|^2 + \bar{p}_1|h_1|^2)\bar{h}_1 g_1. \tag{117}$$

Substituting these equalities into (108) we obtain

$$H_1(0)_{11} = -\frac{\bar{p}_1|g_1|^2 - p_1|h_1|^2}{p_1|g_1|^2 - \bar{p}_1|h_1|^2} \tag{118}$$

$$H_1(-\mathrm{i})_{11} = -\frac{1}{(p_1 + \mathrm{i})(\bar{p}_1 + \mathrm{i})}\frac{\{(1 + \bar{p}_1^2)p_1|g_1|^2 - (1 + p_1^2)\bar{p}_1|h_1|^2\}}{p_1|g_1|^2 - \bar{p}_1|h_1|^2} \tag{119}$$

$$H_1(-\mathrm{i})_{21} = \mathrm{i}\frac{1}{(p_1 + \mathrm{i})(\bar{p}_1 + \mathrm{i})}\frac{(p_1^2 - \bar{p}_1^2)\bar{h}_1 g_1}{\bar{p}_1|g_1|^2 - p_1|h_1|^2}. \tag{120}$$

We write

$$f_1^2 = \mathrm{e}^{-\Theta_1}\,\mathrm{e}^{\mathrm{i}\Phi_1} \tag{121}$$

$$\Phi_1 = 2\kappa_1' x - 2(\kappa_1'\mu_1' - \kappa_1''\mu_1'')t + \Phi_{10} \qquad \Theta_1 = 2\kappa_1''(x - v_1 t - x_1)$$

$$v_1 = \mu_1' + \frac{\kappa_1'}{\kappa_1''}\mu_1'' \tag{122}$$

where superscripts $'$ and $''$ indicate the real and the imaginary parts of a quantity, respectively:

$$p_1 = -\mathrm{i}k_1 \qquad p_1' = k_1'' \qquad p_1'' = -k_1'.$$

We then obtain

$$(S_1)_1 = 1 - 2\frac{(4p_1'^2/|1+p_1^2|^2) + (p_1'^2/p_1''^2)\sin^2\Phi_1}{\cosh^2\Theta_1 + (p_1'^2/p_1''^2)\sin^2\Phi_1} \tag{123}$$

$$\begin{aligned}(S_1)_2 = {}& [2(4p_1'^2/|1+p_1^2|^2)\sinh\Theta_1\cos\Phi_1 + 2(p_1'/p_1'') \\ & \times(1-|p_1|^4/|1+p_1^2|^2)\cosh\Theta_1 + \sin\Phi_1]/[\cosh^2\Theta_1 + (p_1'^2/p_1''^2)\sin^2\Phi_1]^{-1}\end{aligned} \tag{124}$$

$$\begin{aligned}(S_1)_3 = {}& [2(2p_1'(1-|p_1|^2)/|1+p_1^2|^2)\cosh\Theta_1\cos\Phi_1 - 2(2p_1'^2(1+|p_1|^2)/p_1''|1+p_1^2|^2) \\ & \times\sinh\Theta_1\sin\Phi_1]/[\cosh^2\Theta_1 + (p_1'^2/p_1''^2)\sin^2\Phi_1]^{-1}.\end{aligned} \tag{125}$$

These are the expressions of the one-soliton solution for a spin chain with an easy plane which have been found recently [3, 4]. They cannot be obviously factorized in forms of separated variables even in moving coordinates. Hence, it is hopeless to solve the Landau–Lifshitz equation for a spin chain with an easy plane by means of separating variables.

For the limit as $\rho \to 0$, namely, the anisotropy vanishes, it is convenient to use an auxiliary parameter ζ in (6). The parameter k is related with it as

$$k = \mathrm{i}p = \frac{\zeta+\rho}{\zeta-\rho}. \tag{126}$$

One can then express the expression of the one-soliton solution in terms of the parameter ζ. We restrict ζ_1 in the upper half plane of complex ζ and $|\zeta_1| > \rho$. Then from (108), we find

$$p_1' = 2\rho\frac{\zeta_1''}{|\zeta_1-\rho^2|^2} \qquad p_1'' = -\epsilon\frac{|\zeta_1|^2-\rho^2}{|\zeta_1-\rho|^2} \tag{127}$$

where $\epsilon = \pm 1$ correspond to $p_1'' < 0$ and $p_1'' > 0$, respectively. We finally obtain

$$(S_1)_1 = 1 - 2\frac{(\zeta_1''^2/|\zeta_1|^2) + (4\rho^2\zeta_1''^2/(|\zeta_1|^2-\rho^2)^2)\sin^2\Phi_1}{\cosh^2\Theta_1 + (4\rho^2\zeta_1''^2/(|\zeta_1|^2-\rho^2)^2)\sin^2\Phi_1} \tag{128}$$

$$\begin{aligned}(S_1)_2 = {}& [2(\zeta_1''^2/|\zeta_1|^2)\sinh\Theta_1\cos\Phi_1 - 2(\zeta_1'\zeta_1''(|\zeta_1|^2+\rho^2)/|\zeta_1|^2(|\zeta_1|^2-\rho^2)) \\ & \times\cosh\Theta_1\sin\Phi_1]/[\cosh^2\Theta_1 + (4\rho^2\zeta_1''^2/(|\zeta_1|^2-\rho^2)^2)\sin^2\Phi_1]^{-1}\end{aligned} \tag{129}$$

$$\begin{aligned}(S_1)_3 = {}& [2(\zeta_1'\zeta_1''/|\zeta_1|^2)\cosh\Theta_1\cos\Phi_1 + 2(\zeta_1''^2(|\zeta_1|^2+\rho^2)/|\zeta_1|^2(|\zeta_1|^2-\rho^2)) \\ & \times\sinh\Theta_1\sin\Phi_1]/[\cosh^2\Theta_1 + (4\rho^2\zeta_1''^2/(|\zeta_1|^2-\rho^2)^2)\sin^2\Phi_1]^{-1}.\end{aligned} \tag{130}$$

12. Concluding remark

To show the expected asymptotic behaviours of the multi-soliton solutions in the limit of $t \to \pm\infty$, it is convenient to rotate the 1-axis to the 3-axis, as it has been discussed in a previous paper [4].

In the present work, an appropriate procedure is developed for solving the Riemann matrix with poles and it has been used to give explicit expressions of the multi-soliton solutions to the L–L equation for a spin chain with an easy plane. It is obvious that the same procedure can be used for other nonlinear equations if their Lax pairs are 2×2 matrices with anti-Hermitian evolution properties.

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Appendix

We write

$$Q_{nm} = \frac{1}{p_n + \bar{p}_m} \qquad n, m \in \{1, 2, \ldots, M\}. \tag{A.1}$$

Its minor of order of r, $Q_r(n_1, \ldots, n_r; m_1, \ldots, m_r)$, is a determinant of a submatrix of Q by taking the remaining $(n_1, \ldots, n_r)$th columns and $(m_1, \ldots, m_r)$th rows. From (A.1) we can find

$$Q_r(n_1, \ldots, n_r; m_1, \ldots, m_r) = \prod_{n,m} \frac{1}{p_n + \bar{p}_m} \prod_{n<n', m<m'} (p_n - p_{n'})(\bar{p}_m - \bar{p}_{m'}) \tag{A.2}$$

where

$$n, n' \in \{n_1, \ldots, n_r\} \qquad m, m' \in \{m_1, \ldots, m_r\}. \tag{A.3}$$

Hence, Q_M is also obtained by setting $r = M$. In terms of p_m, we have

$$\boldsymbol{a}_j \equiv \mathrm{i}\boldsymbol{a}_j = \frac{1}{p_j + \bar{p}_j} \prod_{n \neq j} \frac{p_j - p_n}{p_j + \bar{p}_n}. \tag{A.4}$$

$\hat{Q}_{M-1}(j; l)$ denotes a minor of order of M, that is a determinant of a submatrix of Q by deleting the jth column and the lth row. Its expression is similar to the right-hand side of (A.2), but replacing (A.3) with

$$m, m' \neq j \qquad n, n' \neq l. \tag{A.5}$$

Elements of Q^{-1} are given by

$$(Q^{-1})_{lj} = (-1)^{j+l} \frac{\hat{Q}_{M-1}(j; l)}{Q_M}. \tag{A.6}$$

We have

$$(Q^{-1})_{lj} = \boldsymbol{a}_j^{-1} Q_{jl} \bar{\boldsymbol{a}}_l^{-1}. \tag{A.7}$$

Introducing a diagonal matrix

$$(\boldsymbol{a}) = \mathrm{diag}(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_M) \tag{A.8}$$

(A.7) can be rewritten in matrix form,

$$Q^{-1} = (\bar{\boldsymbol{a}})^{-1} Q^{\mathrm{T}} (\boldsymbol{a})^{-1}. \tag{A.9}$$

We write

$$\boldsymbol{a}(p) = \prod_{n=1}^{M} \frac{p - p_n}{p + \bar{p}_n}. \tag{A.10}$$

$\boldsymbol{a}(p)$ can be expanded in partial fraction

$$\boldsymbol{a}(p) = 1 - \sum_{n=1}^{M} \frac{1}{p + \bar{p}_n} \frac{1}{\bar{\boldsymbol{a}}_n}. \tag{A.11}$$

Setting $p = p_m$, the left-hand side vanishes obviously, and hence

$$\sum_{n=1}^{M} \frac{1}{p_m + \bar{p}_n} \frac{1}{\bar{a}_n} = 1. \tag{A.12}$$

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